THE PPW CONJECTURE IN CURVED SPACES

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ABSTRACT. In Euclidean ([AB92]) and Hyperbolic ([BL07]) space, and the round hemisphere ([AB01]), geodesic balls maximize the gap $\lambda_2 - \lambda_1$ of Dirichlet eigenvalues, among domains with fixed λ_1. We prove an upper bound on $\lambda_2 - \lambda_1$ for domains in manifolds with certain curvature bounds. The inequality is sharp on geodesic balls in spaceforms.

1. INTRODUCTION

In the '90s Ashbaugh-Benguria [AB92] settled the following conjecture of Payne, Polya and Weinberger.

Theorem 1.1 (PPW conjecture, [AB92]). *Among all bounded domains in $\mathbb{R}^n$, the round ball uniquely maximizes the ratio $\frac{\lambda_2}{\lambda_1}$ of first and second Dirichlet eigenvalues.*

Given a bounded domain $\Omega \subset \mathbb{R}^n$, the Dirichlet eigenvalues $\lambda_i = \lambda_i(\Omega)$ are solutions to the PDE

$$\Delta u + \lambda_i u = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1}$$

where Δ denotes the usual Laplacian $\sum_k \partial_k^2$.

Physically the λ_i correspond to harmonics in a flat drum of shape Ω, so Theorem 1.1 says that one can tell whether a drum is circular by listening to only the first two harmonics. As an aside we mention that Theorem 1.1 is very unstable: by gluing balls of various radii together with thin strips, one can construct domains with ratio λ_2/λ_1 arbitrarily close to the maximum, but which are far from being circular.

Payne-Polya-Weinberger [PPW56] originally bounded the ratio λ_2/λ_1 by 3. Their bound was subsequently improved by Brands [Bra64], de Vries [dV67], then Chiti [Chi83], until Ashbaugh-Benguria proved the sharp inequality, building on the work of Chiti and Talenti [Tal76]. For more history and references see [AB92].

If one considers the problem (1) for domains in a curved space M, with the corresponding metric Laplacian, one is effectively considering harmonics on a drum with tension. Benguria-Linde [BL07] extended the PPW conjecture to hyperbolic space.

Theorem 1.2 (PPW for hyperbolic space, [BL07]). *Among all bounded domains in $\mathbb{H}^n$ with the same fixed first Dirichlet eigenvalue λ_1, the geodesic ball maximizes λ_2.*

In $\mathbb{R}^n$ the ratio λ_2/λ_1 is scale-invariant, but in other spaces the appropriate inequality requires one to normalize competitors by λ_1. Ashbaugh-Benguria [AB01] also extended the PPW conjecture to the hemisphere in S^n.

Theorem 1.3 (PPW for hemispheres, [AB01])**.** *Among all bounded domains in the hemisphere of S^n with the same fixed Dirichlet eigenvalue λ_1, the geodesic ball maximizes λ_2.*

In this paper we seek to prove a general upper bound, in terms of geometric quantities, on the gap $\lambda_2 - \lambda_1$ for a bounded domain in a manifold M, and which reduces to the inequalities 1.1, 1.2, 1.3 when M is a spaceform. The case of warped product manifolds has been considered by Miker [Mik09] in her thesis, though we find her result less geometrically intuitive.

Before stating our Theorem we introduce some notation. Given a Riemannian manifold M^n, we write Sect_M, Ric_M for the sectional, Ricci curvatures (respecively). Given a bounded domain $\Omega \subset M$, write $|\Omega|_M$ for the n-dimensional volume of Ω, $|\partial\Omega|_M$ for the $(n-1)$ Hausdorff measure of $\partial\Omega$, and $\mathrm{diam}(\Omega)$ for the diameter of Ω, each taken with respect to M's Riemannian metric.

Let $N^n(k)$ be the spaceform of constant sectional curvature k. Define the generalized sine function sn_k on $\mathbb{R}$ by

$$\mathrm{sn}_k(r) = \begin{cases} \frac{1}{\sqrt{k}} \sin(\sqrt{k} r) & k > 0 \\ r & k = 0 \\ \frac{1}{\sqrt{-k}} \sinh(\sqrt{-k} r) & k < 0 \end{cases}.$$

The following isoperimetric inequality holds for any bounded domain $\Omega \subset N^n(k)$:

$$|\partial\Omega|_N \geq A_{n,k}(|\Omega|_N), \tag{2}$$

with equality iff Ω is a geodesic ball (see [Sch44]).

Fix (for the duration of this paper) M^n to be a complete, simply-connected n-manifold with $\mathrm{Sect}_M \leq k$. Then for some $\alpha \leq 1$, M satisfies an isoperimetric inequality

$$|\partial\Omega|_M \geq \alpha A_{n,k}(|\Omega|_M) \tag{3}$$

for any bounded domain Ω. We assume throughout this paper that $\alpha > 0$, which is no real loss of generality as we only concern ourselves with a compact neighborhood of Ω.

If $k \leq 0$ then Ω has a closed geodesic convex hull, which we write as $\mathrm{hull}\Omega$. Using elementary comparison geometry one can verify that $\mathrm{diam}(\Omega) = \mathrm{diam}(\mathrm{hull}\Omega)$.

If $k > 0$, we impose the condition on Ω that we can find some strongly convex closed set, which we also write as $\mathrm{hull}\Omega$, containing Ω and satisfying the following properties

A) $\mathrm{diam}\Omega = \mathrm{diam}(\mathrm{hull}\Omega) < \min\{\frac{\pi}{2\sqrt{k}}, \text{injectivity radius of } M\}$,
B) $|\mathrm{hull}\Omega|_M < |N(k)|_{N(k)}/2$.

By strongly convex we mean that the minimizing geodesic connecting any two points in $\mathrm{hull}\Omega$ itself lies in $\mathrm{hull}\Omega$. We require A) so that the exponential function $\exp_p$ is a diffeomorphism onto $\mathrm{hull}\Omega$, for any $p \in \mathrm{hull}\Omega$; we require B) so that we can ultimately work in the hemisphere of N.

We extend Theorems 1.1, 1.2, 1.3 to prove the following inequality for the gap $\lambda_2 - \lambda_1$.

Theorem 1.4. *Let Ω be a bounded domain in M^n. If $k > 0$ let Ω be such that some* $\mathrm{hull}\Omega$ *exists. Let $B_{\alpha,\Omega}$ be a geodesic ball in $N^n(k)$, normalized so that $\lambda_1(B_{\alpha,\Omega}) = \alpha^{-2}\lambda_1(\Omega)$.*

If $\mathrm{Ric} \geq (n-1)K$ *on* $\mathrm{hull}\Omega$*, then*

$$\lambda_2(\Omega) - \lambda_1(\Omega) \leq \left(\frac{\mathrm{sn}_K(\mathrm{diam}\Omega)}{\mathrm{sn}_k(\mathrm{diam}\Omega)}\right)^{2n-2} (\lambda_2(B_{\alpha,\Omega}) - \lambda_1(B_{\alpha,\Omega})). \tag{4}$$

In particular, if $k = K$ *then the constant factor is* 1*, and the inequality is sharp on geodesic balls.*

On spaceforms (i.e. when $k = K$) Theorem 1.4 reduces to the sharp estimates in [AB92], [BL07], [AB01]. In Hadamard manifolds we have a more explicit estimate, due to the scaling of λ_i in $\mathbb{R}^n$.

Corollary 1.5. *Suppose* $k = 0$*, and* Ω *is a bounded domain in* M^n *so that* $\mathrm{Ric}_M \geq (n-1)K$ *on* $\mathrm{hull}\Omega$*. Then*

$$\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} - 1 \leq \frac{1}{\alpha^2}\left(\frac{\sinh(\sqrt{-K}\mathrm{diam}(\Omega))}{\sqrt{-K}\mathrm{diam}(\Omega)}\right)^{2n-2}\left(\frac{\lambda_2(B_1^n)}{\lambda_1(B_1^n)} - 1\right).$$

Here B_1^n *is the unit ball in* $\mathbb{R}^n$*.*

Remark 1.6. The constant factor in Theorem 1.4 is the ratio of areas of geodesics spheres:

$$\lambda_2(\Omega) - \lambda_1(\Omega) \leq \left(\frac{|\partial B_{\mathrm{diam}\Omega}|_{N(K)}}{|\partial B_{\mathrm{diam}\Omega}|_{N(k)}}\right)^2 (\lambda_2(B_{\alpha,\Omega}) - \lambda_1(B_{\alpha,\Omega})).$$

Remark 1.7. We emphasize that in many cases α can be explicitly computed. If $k = 0$, then Croke [Cro84] proved an isoperimetric relation

$$|\partial\Omega| \geq c_n|\Omega|^{\frac{n-1}{n}},$$

where c_n is given by an integral formula of trigonometric functions. If $n = 4$ then in fact c_n is the Euclidean constant, and so $\alpha = 1$.

More generally, the Hadamard conjecture implies that if $k \leq 0$, then $\alpha = 1$. The conjecture is known in the following case: $n = 2$, proved by Weil [Wei26] (for $k = 0$), and Aubin [Aub76] ($k < 0$); $n = 3$, proved by Kleiner [Kle92]; $n = 4$, proved by Croke [Cro84] when $k = 0$. Further, when $n = 4$ and $k < 0$, Kloeckner-Kuperberg [KK13] proved that domains in M which are appropriately "small" (in a quantitative sense) satisfy the Hadamard conjecture. The problem is open for general n.

If the metric g_M is C^0-close to g_N, then α can be written in terms of this bound.

Our approach follows [AB92], though subtleties arise in the presence of non-constant curvature. We "symmetrize" a function defined on M to a function defined on $N(k)$. We prove a version of Chiti's theorem for this notion of symmetrization, which requires a "sharp" form of Faber-Krahn for manifolds satisfying a weak isoperimetric inequality (3). The constant term in Theorem 1.4 essentially results from the fact that symmetrization does not anymore preserve symmetric functions (Proposition 2.3).

We remark that our choice of test functions differ from [AB92] even when $M = N(k) = \mathbb{R}^n$. Unlike [AB92], we ultimately truncate all our functions to $\mathrm{hull}\Omega$, which slightly changes the symmetrizations.

We are not sure whether the diameter or Ricci curvature assumptions are necessary to obtain a gap bound like Theorem 1.4, though they are necessary in our

proof. We mention that Benguria-Linde [BL06] showed that for geodesic balls in hyperbolic space, the ratio λ_2/λ_1 is strictly decreasing in the radius.

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2. Preliminaries

Given $p \in M$, and vectors $v, w \in T_pM$, write $v \cdot w$ for the Riemannian inner product, and $|v| = \sqrt{v \cdot v}$ for the length. $\exp_p : T_pM \to M$ denotes the the usual Riemannian exponential map. If $f : M \to \mathbb{R}$ is differentiable at p, then ∇f is the gradient vector. We write ω_n for the volume of the Euclidean unit ball in $\mathbb{R}^n$.

For the duration of this paper N^n will denote $N^n(k)$. We fix a $q \in N = N(k)$, and write $r_q(x) = \text{dist}_N(x, q)$. Given a function $f : M \to \mathbb{R}_+$, define $\mu_f(t) = |f > t|_M$. As usual we write $\text{spt}f$ for the support of f.

Definition 2.0.1. Take a bounded domain $D \subset M$, and a non-negative integrable $f : D \to \mathbb{R}_+$. Define the *decreasing (resp. increasing) symmetrizations*

$$S^{D,N}f : N \to \mathbb{R}_+, \quad S_{D,N}f : N \to \mathbb{R}_+,$$

by the formulae

$$\begin{aligned} S^{D,N}f(x) &= \mu_f^{-1}(|B_{r_q(x)}(q)|_N), \\ S_{D,N}f(x) &= \mu_f^{-1}(\max\{|D|_M - |B_{r_q(x)}(q)|_N, 0\}). \end{aligned}$$

Let $S^N D$ be the geodesic ball in N centered at q satisfying $|S^N D|_N = |D|_M$.

In casual terms the $S^{D,N}f$ (resp. $S_{D,N}f$) is the decreasing (resp. increasing) function of $r_q(x)$ fixed by the condition

$$|S^{D,N}f > t|_N = |S_{D,N}f > t|_N = |f > t|_M \quad \forall t > 0.$$

Both $\text{spt}S^{D,N}f$ and $\text{spt}S_{D,N}f$ are contained in the closure of $S^N D$.

Remark 2.1. The decreasing symmetrization is actually independent of D, so long as $D \supset \text{spt}f$. In other words, if $D' \supset D$, then $S^{D,N}f \equiv S^{D',N}f$. However in the definition of increasing symmetrization there is an ambiguity without specifying the domain of definition: if $f(x) = 0$, do we count that towards the domain of f or not?

Proposition 2.2. *For any $p \geq 1$, we have*

$$||f||_{L^p(D)} = ||S^{D,N}f||_{L^p(N)} = ||S_{D,N}f||_{L^p(N)}.$$

Proof. By Fubini's theorem,

$$\begin{aligned} \int_D f^p &= p\int_0^\infty t^{p-1}|f > t|_M dt \\ &= p\int_0^\infty t^{p-1}|S^{D,N}f > t|_N dt \\ &= \int_N (S^{D,N}f)^p. \end{aligned}$$

The case of $S_{D,N}f$ is verbatim. $\square$

Take a $p \in M$, and define

$$m_{D,p}(\rho) = |B_\rho(p) \cap D|_M.$$

Similarly, write $m_N(\rho) = |B_\rho(q)|_N$.

Proposition 2.3. *Suppose $f : D \to \mathbb{R}_+$ is a decreasing function of $r_p(x) = \operatorname{dist}_M(x, p)$, then*

$$S^{D,N} f(x) = f((m_{D,p}^{-1} \circ m_N)(r_q(x))).$$

If, on the other hand, f is increasing in r_p, then

$$S_{D,N} f(x) = f((m_{D,p}^{-1} \circ m_N)(r_q(x))).$$

Proof. If f is decreasing in r_p, then $f^{-1}(t, \infty) = B_\rho(p) \cap D$ for some $\rho = \rho(t)$, and so $\mu_f(t) = m_{D,p}(\rho(t))$. Similarly, f is increasing then $f^{-1}(t, \infty) = D \sim \overline{B_\rho(p)}$. Now use the definition of $S^{D,N} f$, $S_{D,N} f$. □

Proposition 2.4. *If $f, g : D \to \mathbb{R}_+$, then*

$$\int_{S^N D} (S^{D,N} f)(S_{D,N} g) \leq \int_D fg \leq \int_{S^N D} (S^{D,N} f)(S^{D,N} g).$$

Proof. By Fubini's theorem, we obtain

$$\begin{aligned} \int_D fg &= \int_0^\infty \int_0^\infty |\{f > s\} \cap \{g > t\}|_M ds dt \\ &\leq \int_0^\infty \int_0^\infty \min\{|f > s|_M, |g > t|_M\} ds dt \\ &= \int_0^\infty \int_0^\infty |\{S^{D,N} f > s\} \cap \{S^{D,N} g > t\}|_N ds dt \\ &= \int_{S^N D} S^{D,N} f S^{D,N} g. \end{aligned}$$

The penultimate equality arises because both $S^{D,N} f$, $S^{D,N} g$ are decreasing functions of r_q (i.e. the upper level-sets are balls concentric about q). By the same logic, since $S_{D,N} g$ is an increasing function of r_q,

$$\begin{aligned} &\int_0^\infty \int_0^\infty |\{f > s\} \cap \{g > t\}|_M ds dt \\ &\geq \int_0^\infty \int_0^\infty \max\{|f > s|_M + |g > t|_M - |D|_M, 0\} ds dt \\ &= \int_0^\infty \int_0^\infty |\{S^{D,N} f > s\} \cap \{S_{D,N} g > t\}|_N ds dt. \end{aligned}$$ □

Proposition 2.5. *For any $\beta > 0$,*

$$S^{D,N}(f^\beta) = (S^{D,N} f)^\beta$$

and similarly for $S_{D,N} f$.

Proof. We have $\mu_{f^\beta}(t^\beta) = \mu_f(t)$, and hence $\mu_{f^\beta}^{-1} = (\mu_f^{-1})^\beta$. □

3. Faber-Krahn and Chiti

We need the following weak version of Faber-Krahn. The inequality (5) is a standard argument, but we find that despite any sharpness of the isoperimetric profile, we can still obtain a characterization of equality. Recall the definition (3) of α.

Theorem 3.1 (weak Faber-Krahn)**.** *If Ω is a bounded domain in M, then*

$$\lambda_1(\Omega) \geq \alpha^2 \lambda_1(S^N\Omega), \tag{5}$$

with equality if and only if

$$S^{\Omega,N} u_1 \equiv v_1$$

where u_1 is the first Dirichlet eigenfunction of Ω, and v_1 the first Dirichlet eigenfunction on $S^N\Omega$, both normalized so that

$$||u_1||_{L^2(\Omega)} = ||v_1||_{L^2(S^N\Omega)}.$$

Proof. Write $S^N\Omega = B = B_R(q)$, and without loss of generality suppose $||u_1||_{L^2(\Omega)} = ||v_1||_{L^2(B)} = 1$, so of course $||S^{\Omega,N}u_1||_{L^2(B)} = 1$ also. Let $\mu(t) = |u_1 > t|_M$. For ease of notation write $A = A_{n,k}$ for the isoperimetric profile (2) of the model space $N^n(k)$, and $\lambda_1 = \lambda_1(\Omega)$.

We have, for a.e. t,

$$\begin{aligned} -\mu'(t) &\geq |\partial\{u_1 > t\}|_M^2 \left(\int_{\{u_1 = t\}} |\nabla u_1| \right)^{-1} \\ &\geq \alpha^2 A(|u_1 > t|_M)^2 \left(\int_{\{u_1 = t\}} |\nabla u_1| \right)^{-1} \\ &= \alpha^2 A(\mu(t))^2 \left(\int_{u_1 > t} -\Delta u_1 \right)^{-1} \\ &= \alpha^2 A(\mu(t))^2 \left(\lambda_1 \int_0^{\mu(t)} \mu^{-1}(\sigma) d\sigma \right)^{-1}, \end{aligned}$$

and hence

$$(\mu^{-1})'(s) \geq -\frac{\lambda_1}{\alpha^2} A^{-2}(s) \int_0^s \mu^{-1}(\sigma) d\sigma.$$

Since $|B|_N = |\Omega|_M$, and $u_1 = 0$ on $\partial\Omega$, then $S^{\Omega,N}u_1$ has Dirichlet boundary conditions. If $S^{\Omega,N}u_1 \not\equiv v_1$, then

$$\lambda_1(S^N\Omega) < \int_B |\nabla S^{\Omega,N} u_1|^2.$$

Write $m(r) = |B_r(q)|_N$, and observe that $A(s) = m'(m^{-1}(s))$. Since $S^{\Omega,N}u_1(r) = \mu^{-1}(m(r))$, we have

$$|\nabla S^{\Omega,N} u_1|^2 = \left[(\mu^{-1})'(m(r)) m'(r) \right]^2.$$

Therefore, we calculate

$$\begin{aligned}
\lambda_1(S^N\Omega) &< \int_B ((\mu^{-1})'(m(r))m'(r))^2 \\
&= \int_0^R ((\mu^{-1})'(m(r))m'(r))^2 m'(r)dr \\
&\leq \frac{\lambda_1}{\alpha^2}\int_0^R \frac{m'(r)^2}{A(m(r))^2}|(\mu^{-1})'|(m(r))\int_0^{m(r)} \mu^{-1}(\sigma)d\sigma m'(r)dr \\
&= \frac{\lambda_1}{\alpha^2}\int_0^R \frac{A(m(r))^2}{A(m(r))^2}|(\mu^{-1})'|(m(r))\int_0^{m(r)} \mu^{-1}(\sigma)d\sigma m'(r)dr \\
&\leq \frac{\lambda_1}{\alpha^2}\int_0^{|B|} ((-\mu^{-1})'(s))\int_0^s \mu^{-1}(\sigma)d\sigma ds \\
&= \frac{\lambda_1}{\alpha^2}\int_0^R \mu^{-1}(m(r))^2 m'(r)dr \\
&= \frac{\lambda_1}{\alpha^2}\int_B (S^N u_1)^2 \\
&= \frac{\lambda_1}{\alpha^2}. \qquad \square
\end{aligned}$$

Suppose $B_{\alpha,\Omega}$ is a ball in N, centered at q, with first eigenvalue $\lambda_1(B_{\alpha,\Omega}) = \lambda_1(\Omega)/\alpha^2$, and first eigenfunction z. By the maximum principle and simplicity of λ_1, z is a decreasing function of r_q. By Faber-Krahn above, $\lambda_1(B_{\alpha,\Omega}) \geq \lambda_1(S^N\Omega)$, and hence $B \subset S^N\Omega$. Further, if $B = S^N\Omega$ then necessarily $z \equiv S^N u_1$.

We obtain the following weak version of Chiti's theorem [Chi83].

Theorem 3.2 (weak Chiti)**.** *Let $\Omega \subset M$ be a bounded domain with first eigenvalue $\lambda_1(\Omega)$, and first eigenfunction u_1. Let $B_{\alpha,\Omega} = B_R(q)$ be a ball in N with first eigenvalue $\lambda_1(B_{\alpha,\Omega}) = \lambda_1(\Omega)/\alpha^2$, and first eigenfunction z. Let u_1 and z be normalized so that*

$$||u_1||_{L^2(\Omega)} = ||z||_{L^2(B_{\alpha,\Omega})}.$$

Then there is an $r_0 \in (0,R)$ so that

$$\begin{aligned} z &\geq S^{\Omega,N}u_1 \text{ on } [0,r_0] \\ z &\leq S^{\Omega,N}u_1 \text{ on } [r_0,R]. \end{aligned}$$

Proof. Let $\mu(t) = |u_1 > t|_M$ and $\nu(t) = |z > t|_N$. Write $\lambda_1 = \lambda_1(\Omega)$. Recall we had

$$(\mu^{-1})'(s) \geq -\frac{\lambda_1}{\alpha^2}A^{-2}(s)\int_0^s \mu^{-1}(\sigma)d\sigma.$$

By repeating the proof of this with ν instead of μ, we obtain

$$\begin{aligned}
(\nu^{-1})'(s) &= -\lambda_1(B_{\alpha,\Omega})A^{-2}(s)\int_0^s \nu^{-1}(\sigma)d\sigma \\
&= -\frac{\lambda_1}{\alpha^2}A^{-2}(s)\int_0^s \nu^{-1}(\sigma)d\sigma.
\end{aligned}$$

The normalization implies $s_0 = \sup\{s \in (0,|B|_N) : \mu^{-1}(s) \leq \nu^{-1}(s)\}$ is defined and positive. If $s_0 = |B|_N$, then since $\nu^{-1}(|B|_N) = 0$ and μ^{-1} is decreasing, we necessarily have that $|B|_N = |\Omega|_M$. Otherwise u_1 would be zero on an open set,

contradicting unique continuation. If $|B|_N = |\Omega|_M$ then by Theorem 3.1 $S^{\Omega,N}u_1 \equiv z$ and the Theorem is vacuous.

So we can assume $s_0 \in (0, |B|_N)$. Clearly $\mu^{-1} \geq \nu^{-1}$ on $[s_0, |B|_N]$, and $\mu^{-1}(s_0) = \nu^{-1}(s_0)$. We show $\mu^{-1} \leq \nu^{-1}$ on $[0, s_0]$.

Suppose, towards a contradiction, that $\beta = \sup_{[0,s_0]} \frac{\mu^{-1}}{\nu^{-1}} > 1$. Then we calculate, for $s \in [0, s_0]$,

$$(\beta\nu^{-1} - \mu^{-1})'(s) \leq -\frac{\lambda_1}{\alpha^2} A^{-2}(s) \int_0^s (\beta\nu^{-1} - \mu^{-1})(\sigma) d\sigma \leq 0.$$

And therefore

$$(\beta\nu^{-1} - \mu^{-1})(s) \geq (\beta\nu^{-1} - \mu^{-1})(s_0) = (\beta - 1)\nu^{-1}(s_0) > 0$$

for any $s \in [0, s_0]$, contradicting our choice of β. The Theorem follows by choosing r_0 which satisfies $|B_{r_0}(q)|_N = s_0$. □

Corollary 3.3. *If $F : S^N\Omega \to \mathbb{R}_+$ is a decreasing function of r_q, then*

$$\int_{S^N\Omega} (S^{\Omega,N}u_1)^2 F \leq \int_{B_{\alpha,\Omega}} z^2 F$$

with $B_{\alpha,\Omega}$, z as in Theorem 3.2. If F is an increasing function of r_q, then

$$\int_{S^N\Omega} (S^{\Omega,N}u_1)^2 F \geq \int_{B_{\alpha,\Omega}} z^2 F.$$

Proof. Let r_0 be as in Theorem 3.2. For F decreasing, we have that

$$(z^2 - (S^{\Omega,N}u_1)^2)(F - F(r_0)) \geq 0,$$

with support in $S^N\Omega$. Therefore we have

$$\int_{S^N\Omega} (z^2 - (S^{\Omega,N}u_1)^2) F \geq F(r_0) \left(\int_B z^2 - \int_{S^N\Omega} (S^{\Omega,N}u_1)^2 \right) = 0$$

having used Proposition 2.2. The case of F increasing follows similarly. □

4. Proof of Theorem

Fix (for the duration of this paper) Ω, $B = B_{\alpha,\Omega}$ as in Theorem 3.2, so that $\lambda_1(B) = \lambda_1(\Omega)/\alpha^2$. Take as before u_1 for the first eigenfunction of Ω, and z the first eigenfunction of B. We will sometimes abbreviate $\lambda_i = \lambda_i(\Omega)$.

If $P : \Omega \to \mathbb{R}$ is any Lipschitz function such that Pu_1 is L^2 orthogonal to u_1, then

$$\int_\Omega |\nabla P|^2 u_1^2 \geq (\lambda_2(\Omega) - \lambda_1(\Omega)) \int_\Omega P^2 u_1^2 \tag{6}$$

by min-max (Pu_1 has the right boundary conditions) and integration by parts. We cook up a collection of good test functions P_i.

Write $r_p(x) = \mathrm{dist}_M(p, x) = |\exp_p^{-1}(x)|$, and define $\sigma(r)$ by the condition

$$|B_{\sigma(r)}(q)|_N = |B_r(p) \cap \mathrm{hull}\Omega|_M.$$

In the notation of Proposition 2.3, $\sigma(r) = (m_N^{-1} \circ m_{\mathrm{hull}\Omega,p})(r)$.

Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-negative Lipschitz function with $h(0) = 0$. For a given $p \in \mathrm{hull}\Omega$, define $P_p : \mathrm{hull}\Omega \to T_pM$ by

$$P_p(x) = \frac{\exp_p^{-1}(x)}{r_p} h(\sigma(r_p)).$$

Lemma 4.1. *We can choose a* $p \in \mathrm{hull}\Omega$ *so that* $\int_\Omega P_p(x) u_1^2(x)dx = 0$.

Proof. Define the vector field

$$X(p) = \int_\Omega P_p u_1^2.$$

We show the integral curves of X define a mapping of hullΩ to itself. Since hullΩ is convex and contained in the injectivity radius, hullΩ is topologically a ball, and therefore X must have a zero by the Brouwer fixed point Theorem.

Take $q \notin \mathrm{hull}\Omega$, but near enough so $\exp_q$ is a diffeomorphism on hullΩ. Let $p \in \mathrm{hull}\Omega$ be the nearest point to p. By convexity, the vector $\exp_p^{-1}(q)$ defines a supporting hyperplane for hullΩ at p. In other words,

$$\exp_p^{-1}(\mathrm{hull}\Omega) \subset \{v : v \cdot \exp_p^{-1}(q) \leq 0\}.$$

By definition of P, we deduce $X(p) \cdot \exp_p^{-1}(q) \leq 0$ also.

Let $\phi_t(p)$ be the integral curves of $X(p)$, and define the function

$$f(q) = \begin{cases} \mathrm{dist}(q, \mathrm{hull}\Omega) & q \notin \mathrm{hull}\Omega \\ 0 & \text{else} \end{cases}.$$

Since X is Lipschitz we have by the above reasoning that

$$\limsup_{t \to 0_+} \frac{f(\phi_t(p)) - f(p)}{t} \leq Cf(p),$$

and therefore $f(\phi_t(p)) = 0$ if $f(p) = 0$. This shows ϕ_t maps hullΩ into itself. $\square$

Choose an orthonormal basis $\{e_i\}$ of T_pM. Define

$$P_i(x) = e_i \cdot P_p(x),$$

where we choose and fix p (as a function of h) as in Lemma 4.1. So $\int_\Omega P_i u_1^2 = 0$ for each i, and by (6) we have

$$\int_\Omega (\sum_i |\nabla P_i|^2) u_1^2 \geq (\lambda_2 - \lambda_1) \int_\Omega (\sum_i P_i^2) u_1^2 = (\lambda_2 - \lambda_1) \int_\Omega h^2(\sigma(r_p)) u_1^2.$$

For ease of notation, in the following we will write $g \equiv h \circ \sigma$ and $r \equiv r_p$, so that $P_i(x) = e_i \cdot \exp_p^{-1}(x) g(r)/r$. We calculate

$$\begin{aligned} \frac{d}{ds}|_{s=0} P_i(\exp_p(v + sw)) &= \frac{d}{ds}|_{s=0} \left(e_i \cdot (v + sw) \frac{g(|v + sw|)}{|v + sw|} \right) \\ &= e_i \cdot w \frac{g(|v|)}{|v|} + \frac{(e_i \cdot v)(v \cdot w)}{|v|} \frac{d}{dr}|_{r=|v|} \frac{g(r)}{r}. \end{aligned}$$

Choose an orthonormal basis E_i at a fixed $x = \exp_p(v)$, such that $E_1 = \frac{\partial}{\partial r}$. Write

$$w_j = (D \exp_p |_v)^{-1}(E_j)$$

and since $D\exp_p$ is a radial isometry $w_1 = \frac{v}{|v|}$. We have

$$\begin{aligned} E_1 P_i &= e_i \cdot v \frac{g(r)}{r^2} + e_i \cdot v \left(\frac{g(r)}{r} \right)' \\ E_j P_i &= e_i \cdot w_j \frac{g(r)}{r} \qquad j > 1. \end{aligned}$$

Therefore

$$\begin{aligned}\sum_i |\nabla P_i|^2 &= \sum_i (E_1 P_i)^2 + \sum_{j>1,i} (E_j P_i)^2 \\ &= r^2 \left[\frac{g(r)^2}{r^4} + 2\frac{g(r)}{r^2}\left(\frac{g(r)}{r}\right)' + \left(\frac{g(r)}{r}\right)'^2 \right] + \sum_{j>1} |w_j|^2 \frac{g(r)^2}{r^2} \\ &= g'(r)^2 + \frac{g(r)^2}{r^2}\sum_{j>1}|w_j|^2 \\ &\leq g'(r)^2 + \frac{n-1}{\mathrm{sn}_k^2(r)} g(r)^2\end{aligned}$$

having used Rauch's theorem to deduce

$$1 = |D\exp_p|_v(w_j)| \geq \frac{\mathrm{sn}_k(|v|)}{|v|}|w_j|.$$

Recalling the definition $g = h\circ\sigma$, we estimate for a.e. $r \in r_p(\Omega)$,

$$\begin{aligned}g'(r)^2 + \frac{n-1}{\mathrm{sn}_k^2 r} g(r)^2 &= h'(\sigma(r))^2\sigma'(r)^2 + \frac{n-1}{\mathrm{sn}_k^2 r} h(\sigma(r))^2 \\ &\leq C_1^2\left(h'(\sigma(r))^2 + \frac{n-1}{\mathrm{sn}_k^2 \sigma(r)} h(\sigma(r))^2 \right)\end{aligned}$$

where

$$C_1 = \max_{r\in r_p(\Omega)} \left\{ \sigma'(r), \frac{\mathrm{sn}_k(\sigma(r))}{\mathrm{sn}_k(r)} \right\}. \tag{7}$$

We obtain

Theorem 4.2. *For any Lipschitz $h : \mathbb{R}_+ \to \mathbb{R}_+$ with $h(0) = 0$, we can choose a point $p \in \mathrm{hull}\Omega$ so that*

$$(\lambda_2(\Omega) - \lambda_1(\Omega)) \int_\Omega u_1^2 h(\sigma(r_p))^2 \leq C_1^2 \int_\Omega u_1^2 F(\sigma(r_p)).$$

Here $F(t) = h'(t)^2 + \frac{n-1}{\mathrm{sn}_k^2(t)} h(t)^2$, and C_1 as in (7).

Corollary 4.3. *If h, p are as in Theorem 4.2, and h further satisfies:*

$$(\star)\left\{\begin{array}{l} h(r) \textit{ is increasing} \\ F(r) \textit{ is decreasing}\end{array}\right. ,$$

then

$$(\lambda_2(\Omega) - \lambda_1(\Omega)) \int_B z^2 h(r_q)^2 \leq C_1^2 \int_B z^2 F(r_q).$$

Here B and z are as in Theorem 3.2.

Remark 4.4. In Corollary 4.3 we have still not used the lower Ricci curvature bound.

Proof. Extend u_1 by 0 to be define on hullΩ, and recall that Remark 2.1 implies

$$S^{\mathrm{hull}\Omega,N} u_1 \equiv S^{\Omega,N} u_1. \tag{8}$$

We calculate

$$\begin{aligned}\int_\Omega u_1^2 F(\sigma(r_p)) &\le \int_{S^N \mathrm{hull}\Omega} (S^{\mathrm{hull}\Omega,N} u_1)^2 (S^{\mathrm{hull},N}(F\circ\sigma\circ r_p))\\ &= \int_{S^N\Omega} (S^{\Omega,N}u_1)^2 F(r_q)\\ &\le \int_B z^2 F(r_q).\end{aligned}$$

In the first line we used Proposition 2.4; in the second line we used Proposition 2.3, the definition of $\sigma(r)$, and (8); in the third we used Corollary 3.3.

Using the same Theorems in the same order, but since h is increasing, we have

$$\begin{aligned}\int_\Omega u_1^2 h(\sigma(r_p))^2 &\ge \int_{S^N \mathrm{hull}\Omega} (S^{\mathrm{hull}\Omega,N} u_1)^2 (S_{\mathrm{hull},N}(h\circ\sigma\circ r_p))^2\\ &= \int_{S^N\Omega} (S^{\Omega,N}u_1)^2 h(r_q)^2\\ &\ge \int_B z^2 h(r_q)^2.\end{aligned}$$

Now plug these calculations into Theorem 4.2. □

Proof of Theorem 1.4. Recall that $B_{\alpha,\Omega} = B_R(q)$ was the geodesic ball in $N^n(k)$ with first eigenvalue $\lambda_1(B_{\alpha,\Omega}) = \lambda_1(\Omega)/\alpha^2$, and $z = z(r_q)$ was its first eigenfunction. Let $J = J(r_q)$ be the radial component of the second Dirchlet eigenfunction of B (c.f. equation 2.11 of [AB92], section 3 of [BL07], section 3 of [AB01]).

Notice that when $k > 0$, the assumption $|\mathrm{hull}\Omega|_M < |N|_N/2$ implies $S^N\Omega \supset B$ lies in the hemisphere.

Define

$$h(t) = \begin{cases} \frac{J(t)}{z(t)} & t\in[0,R)\\ \lim_{s\to R_-} w(s) & t \ge R\end{cases}$$

Using Corollary 3.4 of [AB92] (if $k = 0$), Lemma 7.1 in [BL07] (if $k < 0$), or Theorem 4.1 in [AB01] (if $k > 0$), we deduce that $h(t)$ is increasing, and $F(t) = h'(t)^2 + \frac{n-1}{\mathrm{sn}_k^2(t)} h(t)^2$ is decreasing.

We can therefore apply Theorem 4.3 to deduce

$$(\lambda_2(\Omega) - \lambda_1(\Omega)) \le C_1^2(\lambda_2(B_{\alpha,\Omega}) - \lambda_1(B_{\alpha,\Omega})),$$

with C_1 as in (7).

We show that

$$C_1 \le \frac{|\partial B_{\mathrm{diam}\Omega}|_{N(K)}}{|\partial B_{\mathrm{diam}\Omega}|_{N(k)}}.$$

For ease of notation write $m_\ell(r) = |B_r|_{N(\ell)}$. All balls in M are centered at p, and balls in $N(k)$, $N(K)$ are centered at q, $\tilde q$ (resp.).

Suppose C_p is a geodesic cone in M, centered at p, with solid angle $\gamma n\omega_n$ in T_pM. If $\mathrm{Ric}_M \ge (n-1)K$ on $B_r \cap C_p$, then by the Bishop-Gromov volume comparison we have

$$|\partial B_r \cap C_p|_M \le \gamma |\partial B_r|_{N(K)}.$$

Conversely, choosing a linear isometry $\iota : T_pM \to T_qN(k)$, take

$$C_p' = (\exp_q^{N(k)} \circ \iota \circ (\exp_p^M)^{-1})(C_p)$$

to be a geodesic cone in $N(k)$ with the same cone angle as C_p. Since $\mathrm{Sect}_M \leq k$ we have by Hessian comparision that

$$|B_r \cap C_p|_M \geq |B_r \cap C_p'|_{N(k)} = \gamma |B_r|_{N(k)}.$$

Recall that $\sigma(r) = m_k^{-1}(|B_r(p) \cap \mathrm{hull}\Omega|_M)$. Notice that

$$B_r(p) \cap \mathrm{hull}\Omega \supset B_r(p) \cap C_p$$

where C_p is a geodesic cone at p over $\partial B_r(p) \cap \mathrm{hull}\Omega$. Therefore

$$\begin{aligned}
\sigma'(r) &= \frac{1}{m_k'(m_k^{-1}(|B_r \cap \mathrm{hull}\Omega|_M))} |\partial B_r \cap \mathrm{hull}\Omega|_M \\
&\leq \frac{1}{m_k'(m_k^{-1}(|B_r \cap C_p|_M))} |\partial B_r \cap C_p|_M \\
&\leq \frac{1}{m_k'(m_k^{-1}(\gamma |B_r|_N))} \gamma |\partial B_r|_{N(K)} \\
&\leq \frac{|\partial B_r|_{N(K)}}{|\partial B_r|_{N(k)}}.
\end{aligned}$$

The last inequality follows because the isoperimetric profile $A_{n,k}(s) = m_k'(m_k^{-1}(s))$ is concave. We elaborate. The last inequality is equivalent to

$$m_k'(m_k^{-1}(s)) \leq \frac{m_k'(m_k^{-1}(\gamma s))}{\gamma}$$

for any $\gamma \in (0, 1]$. But the RHS is a dilation of the graph of the LHS, hence the inequality follows if the graph is concave. We calculate

$$(m_k' \circ m_k^{-1})'' = \frac{(m_k' \circ m_k^{-1})(m_k''' \circ m_k^{-1}) - (m_k'' \circ m_k^{-1})^2}{(m_k' \circ m_k^{-1})^3}.$$

Since

$$(m_k' m_k''' - (m_k'')^2)(r) = -(n-1)n^2 \omega_n^2 \mathrm{sn}_k(r)^{2n-4} \leq 0,$$

the graph is concave (here again we use that $S^N\Omega$ lies in the hemisphere of $N(k)$, if $k > 0$).

We prove now the inequality

$$\frac{\mathrm{sn}_k(\sigma(r))}{\mathrm{sn}_k(r)} \leq \frac{|\partial B_r|_{N(K)}}{|\partial B_r|_{N(k)}}.$$

Since $\sigma(r) \leq m_k^{-1}(m_K(r))$, it suffices to prove the inequality

$$m_K(r) \leq m_k \left[\mathrm{sn}_k^{-1} \left(\frac{m_K'(r)}{m_k'(r)} \mathrm{sn}_k(r) \right) \right].$$

We therefore calculate

$$\begin{aligned}
m_k\left[\mathrm{sn}_k^{-1}\left(\frac{m_K'(r)}{m_k'(r)}\mathrm{sn}_k(r)\right)\right] &= m_k\left[\mathrm{sn}_k^{-1}\left(\mathrm{sn}_K(r)\left(\frac{\mathrm{sn}_K(r)}{\mathrm{sn}_k(r)}\right)^{n-2}\right)\right] \\
&\geq m_k\left[\mathrm{sn}_k^{-1}(\mathrm{sn}_K(r))\right] \\
&= n\omega_n\int_0^{\mathrm{sn}_k^{-1}(\mathrm{sn}_K(r))}\mathrm{sn}_k(\rho)^{n-1}d\rho \\
&= n\omega_n\int_0^r \mathrm{sn}_K(\rho)^{n-1}\sqrt{\frac{1-K\mathrm{sn}_K(\rho)^2}{1-k\mathrm{sn}_K(\rho)^2}}d\rho \\
&\geq n\omega_n\int_0^r \mathrm{sn}_K(\rho)^{n-1}d\rho \\
&= m_K(r),
\end{aligned}$$

using that $\mathrm{sn}_k'(r)^2 = 1 - k\mathrm{sn}_k(r)^2$. □

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